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Ascending Chromatic Decomposition of Graphs

Ms.A.Nageswari

Assistant Professor, Department of Mathematics, Sree Saraswathi Thyagaraja College, Pollachi, India.
Corresponding Author Mail Id: nageswari8943@gmail.com

Abstract

In this paper, we combine decomposition and domination and introduce the concept Ascending Chromatic Decomposition ($A\chi'D$) of a graph G. An $A\chi'D$ of a graph G is a collection $\psi = \{G_1, G_2, \dots, G_n\}$ of subgraphs of G such that, each G_i is connected, every edge of G is in exactly one G_i and $\chi'(G_i) = i, 1 \le i \le n$. Ascending Chromatic decomposition number of any graph$G$ is the minimum number of decomposition in which $A\chi'D$ exists and is denoted by $AD\chi'(G)$. In this paper, we find Ascending chromatic decomposition number for $K_n, P_n, C_n, K_{m,n}$ .We also establish the necessary and sufficient condition for Ascending chromatic decomposition number exists in $K_{m,n}$ .We also find Ascending chromatic decomposition number for the corona of path, cycle.

Keywords: chromatic index, decomposition, ascending chromatic decomposition.

AMS Subject Classification (2010): 05C15, 05C70.

Introduction

By a graph, we mean a finite, undirected, non-trivial, connected graph without loops and multiple edges. The order and size of a graph are denoted by p and q respectively. For terms not defined here we refer to Harary(1969) .

Definition 1.1

The corona $G_1 \odot G_2$ of two graphs G_1 and G_2 is defined as the graph obtained by taking one copy of G_1 (with p_1 vertices) and p_1 copies of G_2 and then joining the i^{th} vertex of G_1 to all the vertices in the i^{th} copy of G_2 . In particular, the graph $G \odot K_1$ is denoted by G^+ .

The graph P_n^+ is called comb and the graph C_n^+ is called a crown.

The chromatic index $\chi'(G)$ of a graph G is the minimum number of colors required to color the edges of G so that no two adjacent edges get the same color.

Another important area of graph theory is decomposition of graphs (Juraj Bosak 1990). A decomposition of a graph G is a collection ψ of edge disjoint subgraphs $G_1, G_2, \dots, G_n$ of G such that every edge of G is in exactly one G_i . If each G_i is isomorphic to a subgraph H of G, then ψ is called a H -decomposition. Several authors studied various types of decompositions by imposing conditions on G_i in the decomposition.

We introduce a concept called Ascending Chromatic Decomposition $(A\chi'D)$ of a graph which is motivated by the concepts of Ascending Domination Decomposition (ADD), Ascending Subgraph Decomposition (ASD) and Continuous Monotonic Decomposition (CMD) of a graph. The concept of Ascending Subgraph Decomposition was introduced by Alavi *et al.*, 1987.

Definition 1.2 (Lakshmiprabha and Nagarajan, 2014)

An ADD of a graph G is a collection $\psi = \{G_1, G_2, \dots, G_n\}$ of subgraphs of G such that,

(i) each G_i is connected

(ii) every edge of G is in exactly one G_i

(iii) $\gamma(G_i) = i, 1 \leq i \leq n$.

Definition 1.3

A decomposition of G into subgraphs G_i (not necessarily connected) such that $|E(G_i)| = i$ and G_i is isomorphic to a proper subgroup of G_{i+1}, is called an Ascending Subgraph Decomposition (ASD). (Alavi *et al.*, 1987).

Definition 1.3

A decomposition $\{G_1, G_2, \dots, G_n\}$ of G is said to be a Continuous Monotonic Decomposition (CMD) if each G_i is connected and $|E(G_i)| = i$ for each $i = 1,2, \dots, n$

The concept of continuous monotonic decomposition was introduced by N. Gnana Dhas (Gnana Dhas and Paulraj Joseph, 2000).

In this paper, we initiate a study on $\chi'D$.

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Main Results

We define AscendingChromatic Decomposition $(A\chi'D)$ as follows.

Definition 2

An $A\chi'D$ of a graph G is a collection $\psi = \{G_1, G_2, \dots, G_n\}$ of subgraphs of G such that,

(i) each G_i is connected

(ii) every edge of G is in exactly one G_i

(iii) $\chi'(G_i) = i, 1 \leq i \leq n.$

If a graph G has an $A\chi'D$, we say that G admits Ascending Chromatic Decomposition.

Example 2.2: An $A\chi'D$ $\{G_1, G_2, G_3\}$ of a given graph G is given in Figure 1.

Note that $\chi'(G_i) = i, 1 \leq i \leq 3.$

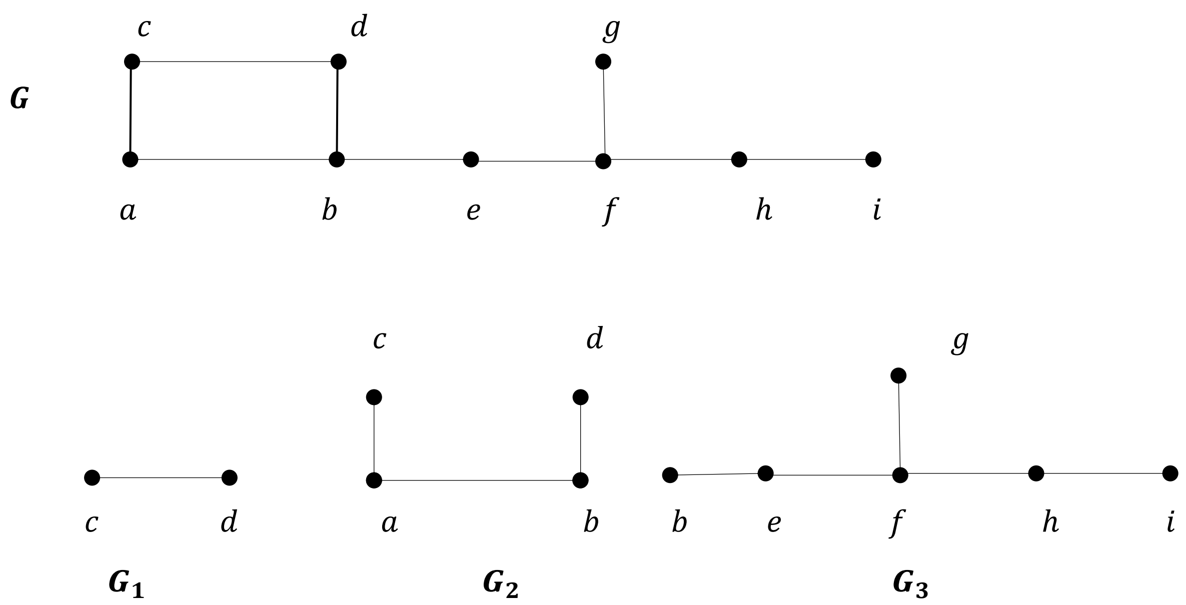

Figure 1: *An* $A\chi'D$ $\{G_1, G_2, G_3\}$*of* G .

Definition 2.3

Ascending Chromatic decomposition number of any graph G is the minimum number of decomposition in which $A\chi'D$ exists and is denoted by $AD\chi'(G)$.

Theorem 2.4. $AD\chi'(P_n) = 2$

Proof

Let $P_n = v_1 v_2 v_3 \dots v_n$ be a path.

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Suppose P_n admit $A\chi'D$

Let $\psi = \{G_1, G_2, \ldots, G_n\}$

Let $G_1 = \{v_1v_2\}$

$G_2 = \{v_2v_3, v_3v_4, \ldots, v_{n-1}v_n\}$

Since $G_1 \cong K_2$, $\chi'(G_1) = 1$

Since $G_2 \cong P_{n-1}$, $\chi'(G_2) = 2$

$A\chi'D$ exists for P_n in which the decomposition is minimum.

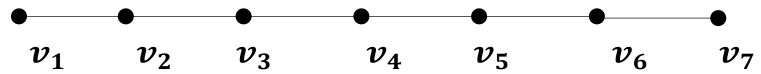
$AD\chi'(P_n) = 2$

Example 2.5

An $A\chi'D$ $\{G_1, G_2\}$ of a given graph P_7 is given in Figure 2.

Note that $\chi'(G_i) = i$ for $i = 1,2$.

P_7



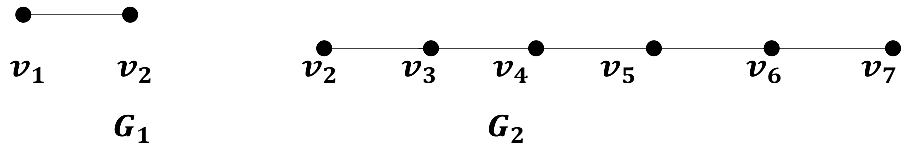


Figure 2: *An* $A\chi'D = \{G_1, G_2\}$ *of* P_7 .

Theorem 2.6: $AD\chi'(C_n) = 2$

Proof

Let $P_n = v_1v_2v_3 \ldots v_n$ be a cycle.

Suppose C_n admit $A\chi'D$

Let $\psi = \{G_1, G_2, \ldots, G_n\}$

Let $G_1 = \{v_1v_n\}$

$G_2 = \{v_1v_2, v_2v_3, v_3v_4, \ldots, v_{n-1}v_n\}$

Since $G_1 \cong K_2$, $\chi'(G_1) = 1$

Since $G_2 \cong P_n$, $\chi'(G_2) = 2$

$A\chi'D$ exists for C_n in which the decomposition is minimum.

$AD\chi'(C_n) = 2$

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Example 2.7

An $A\chi'D$ $\{G_1, G_2\}$ of a given graph C_5 is given in Figure 3.

Note that $\chi'(G_i) = i$ for $i = 1,2$.

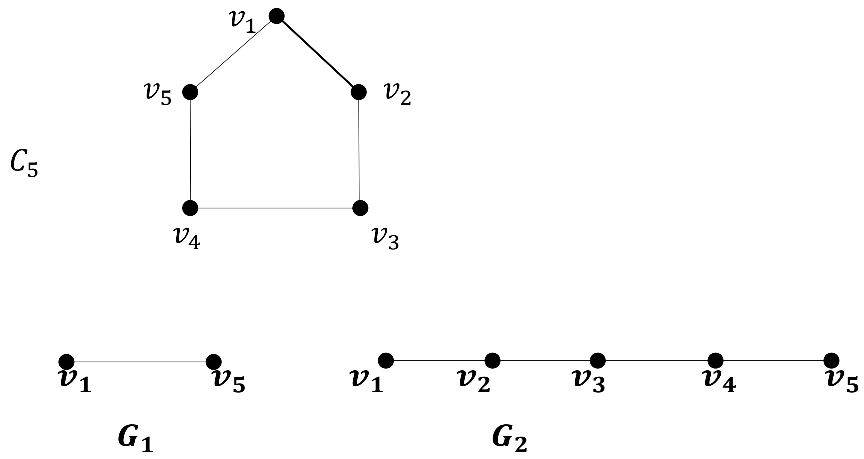

Figure 3: *An* $A\chi'D = \{G_1, G_2\}$ *of* C_5 .

Theorem 2.8. $AD\chi'(P_n^+) = 3$ where $P_n \odot K_1$

Proof

Let $P_n = v_1v_2v_3 \dots v_n$ be a path.

If we attach the vertices $v_1', v_2', \dots, v_n'$ to $v_1, v_2, \dots, v_n$ respectively, then we get $P_n^+ = P_n \odot K_1$

Suppose P_n^+ admit $A\chi'D$

Let $\psi = \{G_1, G_2, \dots, G_n\}$

Let $G_1 = \{v_1v_1'\}$

$G_2 = \{v_1v_2, v_2v_2'\}$

$G_3 = \{remaining\ edges\}$

Since $G_1 \cong K_2$, $\chi'(G_1) = 1$

Since $G_2 \cong P_2$, $\chi'(G_2) = 2$

In $G_3, \Delta(G_3) = 3$

$\chi'(G_3) = 3$

$A\chi'D$ exists for P_n^+ in which the decomposition is minimum.

$AD\chi'(P_n^+) = 3$

Example 2.9

An $A\chi'D$ $\{G_1, G_2, G_3\}$ of a given graph P_5^+ is given in Figure 4.

Note that $\chi'(G_i) = i$ for $i = 1,2,3$.

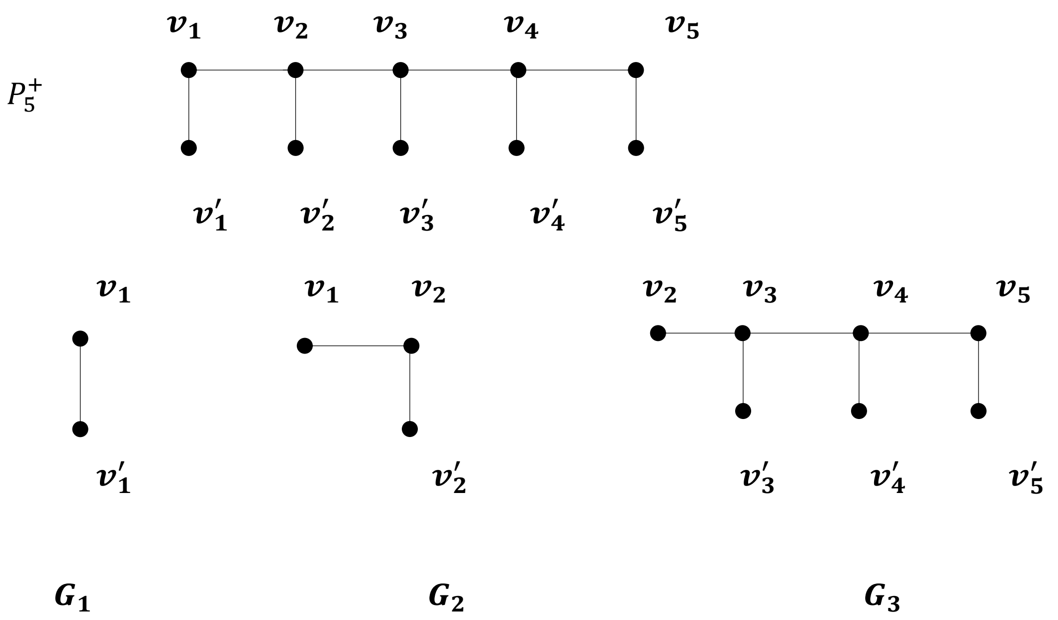

Figure 4: *An* $A\chi'D = \{G_1, G_2, G_3\}$ *of* P_5^+

Theorem 2.10: $AD\chi'(C_n^+) = 3$ where $C_n \odot K_1$

Proof

Let $C_n = v_1v_2v_3 \ldots v_n$ be a path.

If we attach the vertices $v_1', v_2', \ldots, v_n'$ to $v_1, v_2, \ldots, v_n$ respectively, then we get $C_n^+ = C_n \odot K_1$

Suppose C_n^+ admit $A\chi'D$

Let $\psi = \{G_1, G_2, \ldots, G_n\}$

Let $G_1 = \{v_1v_1'\}$

$G_2 = \{v_1v_2, v_2v_2'\}$

$G_3 = \{remaining\ edges\}$

Since $G_1 \cong K_2, \chi'(G_1) = 1$

Since $G_2 \cong P_2, \chi'(G_2) = 2$

In $G_3, \Delta(G_3) = 3$

$\chi'(G_3) = 3$

$A\chi'D$ exists for C_n^+ in which the decomposition is minimum.

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$AD\chi'(C_n^+) = 3$

Example 2.11

An $A\chi'D$ $\{G_1, G_2, G_3\}$ of a given graph C_5^+ is given in Figure 5.

Note that $\chi'(G_i) = i$ for $i = 1,2,3$.

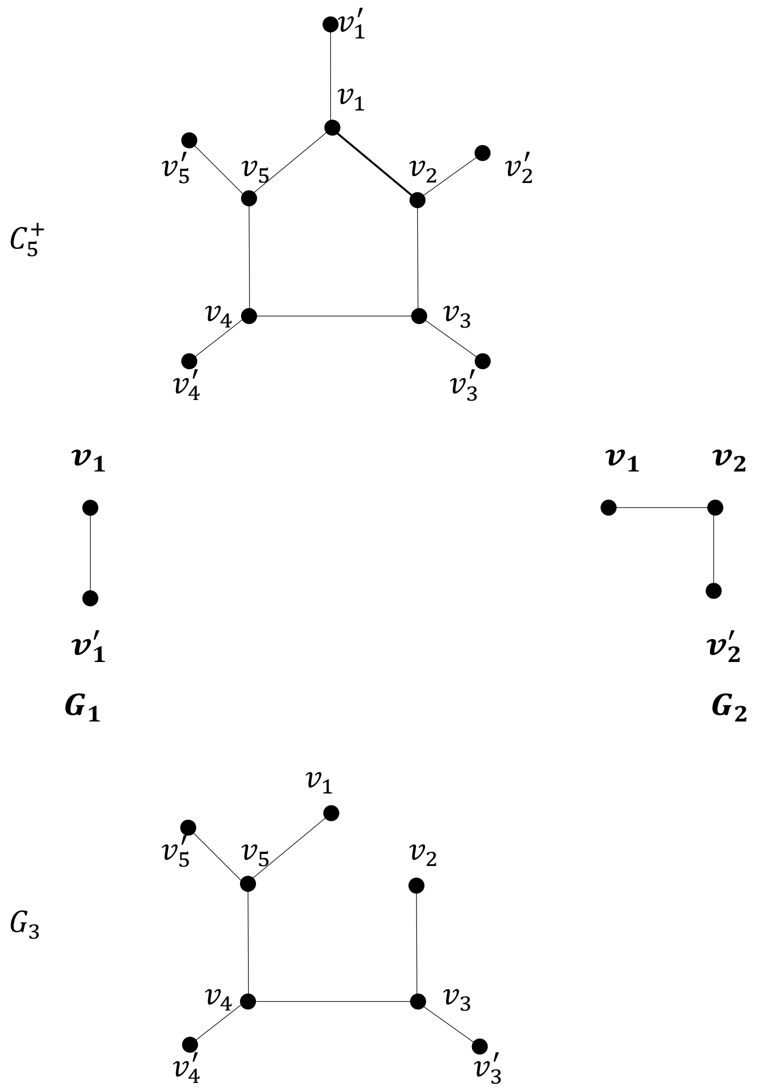

Figure 5: *An* $A\chi'D = \{G_1, G_2, G_3\}$ *of* C_5^+ .

Theorem 2.12: $AD\chi'(K_{n,n}) = n - 1$

Proof:

Let $V = X \cup Y$ be a bipartition of $K_{n,n}$ with $|X| = n$ and $|Y| = n$.

Let $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$

Let $G_1 = \{x_1 y_n\} \cong K_2$

$G_2 = x_1 y_1 x_2 y_2 \dots x_{n-1} y_n x_n \cong P_{2n}$

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$G_3 \cong K_{1,3}$

$G_4 \cong K_{1,4}$

$\vdots \quad \vdots$

$G_{n-2} \cong K_{1,n-2}$

$G_{n-1} = \{\, remaining\ edges\}$

We observe that $\chi'(G_i)$=i for i=1,2,... $n-1$

$A\chi'D$ exists for $K_{n,n}$ in which the decomposition is minimum.

$$AD\chi'(K_{n,n}) = n - 1$$

Example 2.13: An $A\chi'D$ $\{G_1, G_2, G_3\}$ of a given graph $K_{4,4}$ is given in Figure 6.

Note that $\chi'(G_i) = i$ for $i = 1,2,3$.

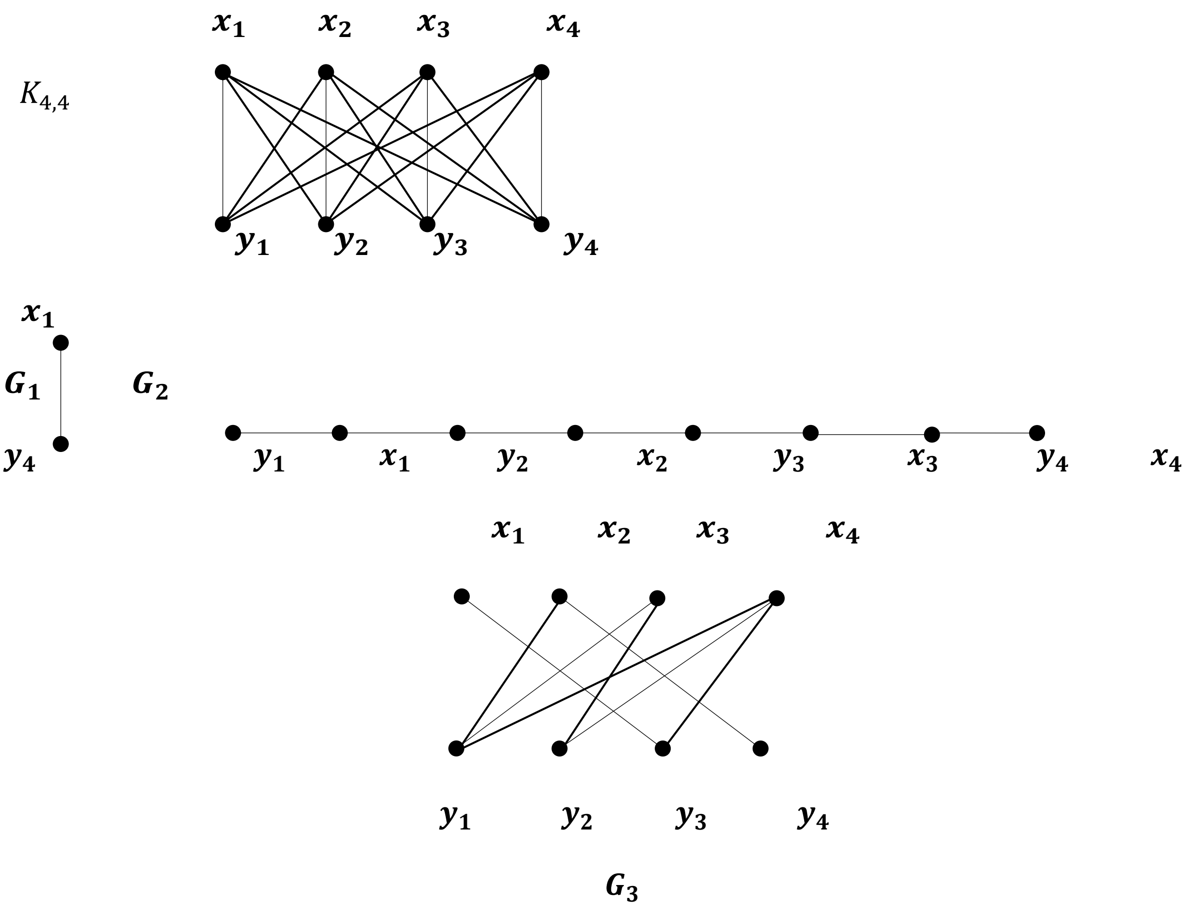

Figure 6: *An* $A\chi'D = \{G_1, G_2, G_3\}$ *of* $K_{4,4}$.

Theorem 2.14.: $AD\chi'(\mathrm{K}_{1,\mathrm{m}}) =$ n iff $(2n+1)^2 = 8p - 7$ where p, the number of vertices.

Proof

Let v be the central vertex and $v_1, v_2, \dots, v_m$ be the pendant vertices of $K_{1,m}$

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Suppose $AD\chi'(K_{1,m}) = n$

Let $G_1 = \{vv_1\}$

$G_2 = \{vv_2, vv_3\}$

$G_3 = \{vv_4, vv_5, vv_6\}$

$\vdots \quad \vdots$

$G_n = \{vv_l, vv_{l+1}, \dots, vv_n\}$

We observe that $\chi'(G_i)$=i for i=1,2,... n and $K_{1,m}$ has $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ edges.

We know that $K_{1,m}$ has $p - 1$ edges where p, the number of vertices in $K_{1,m}$.

$$\text{Therefore, } p - 1 = \frac{n(n+1)}{2}$$

$$\Rightarrow 2(p-1) = n^2 + n$$

$$\Rightarrow n^2 + n - 2(p-1) = 0$$

$$\Rightarrow n = \frac{-1 \pm \sqrt{1+8(p-1)}}{2}$$

$$\Rightarrow 2n + 1 = \pm\sqrt{8p-7}$$

$$\Rightarrow (2n+1)^2 = 8p - 7$$

Retracing the above steps, we get the converse part.

Example 2.15

An $A\chi'D$ $\{G_1, G_2, G_3\}$ of a given graph $K_{1,6}$ is given in Figure 7.

Note that $\chi'(G_i) = i$ for $i = 1,2,3$.

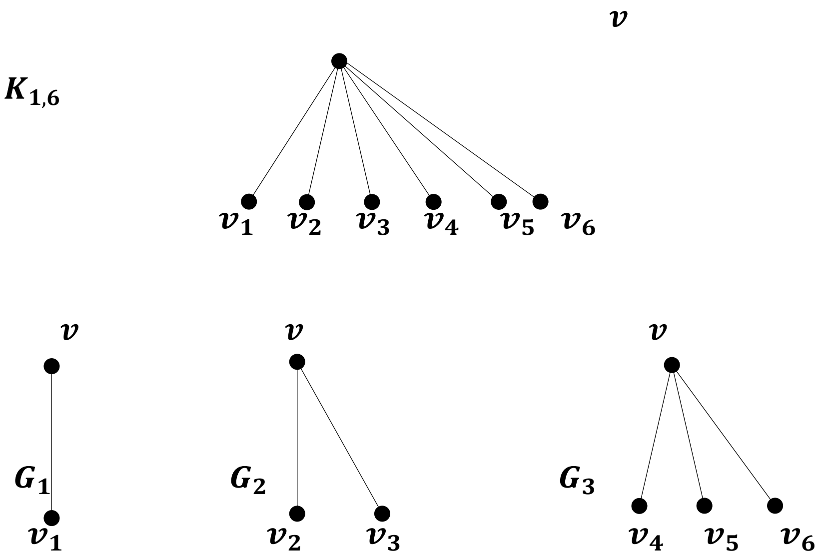

Figure 7: *An* $A\chi'D = \{G_1, G_2, G_3\}$ *of* $K_{1,6}$

Theorem 2.16: $AD\chi'(\mathrm{K}_{\mathrm{m,n}}) = \mathrm{s}, \mathrm{m} \neq n, 1$ iff $(2s+1)^2 = 1 + 8mn$

Proof:

Let $V = X \cup Y$ be a bipartition of $K_{m,n}$ with $|X| = m$ and $|Y| = n$.

Let $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$

Suppose $AD\chi'(\mathrm{K}_{\mathrm{m,n}}) = \mathrm{s}, \mathrm{m} \neq n, 1$

Let $G_1 \cong K_2$

$G_2 \cong K_{1,2}$

$G_3 \cong K_{1,3}$

$\vdots \quad \vdots$

$G_s \cong K_{1,s}$

We observe that $\chi'(G_i)$=i for i=1,2,... s and $K_{m,n}$has $1 + 2 + \cdots + s = \frac{s(s+1)}{2}$ edges.

We know that $K_{m,n}$has mn edges.

Therefore, $mn = \frac{s(s+1)}{2}$

$\Rightarrow 2mn = s^2 + s$

$\Rightarrow s^2 + s - 2mn = 0$

$\Rightarrow s = \frac{-1 \pm \sqrt{1+8mn}}{2}$

$\Rightarrow 2s + 1 = \pm\sqrt{1 + 8mn}$

$\Rightarrow (2s+1)^2 = 1 + 8mn$

Retracing the above steps, we get the converse part.

Example 2.17

An $A\chi'D$ $\{G_1, G_2, G_3\}$ of a given graph $K_{2,3}$ is given in Figure 8.

Note that $\chi'(G_i) = i$ for $i = 1,2,3$.

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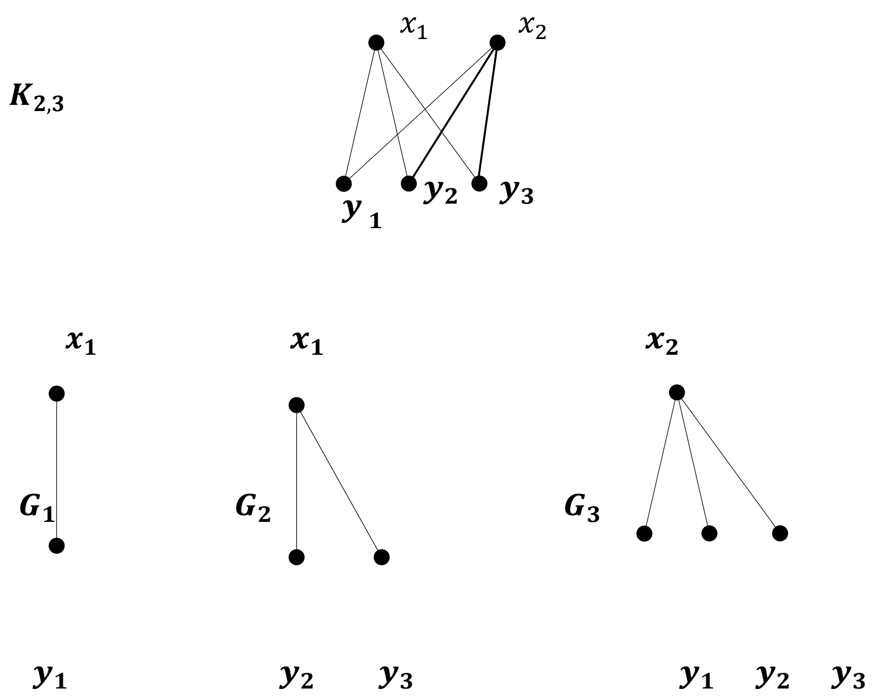

Figure 8: $An\ A\chi'D = \{G_1, G_2, G_3\}\ of\ K_{2,3}$

Theorem 2.18: $AD\chi'(\mathrm{K_n}) = \begin{cases} \frac{n}{2}\ if\ n\ is\ even \\ \frac{n+1}{2}\ if\ n\ is\ odd \end{cases}$

Proof:

Complete graph K_n in which any two distinct vertices are adjacent.

Case i) When n is even

First we take $G_1 = \{v_1 v_{n-2}\} \cong K_2$

$$G_2 \cong C_n$$

$$G_3 \cong K_{1,3}$$

$$G_4 \cong K_{1,4}$$

$$\vdots \quad \vdots$$

$$G_{\frac{n}{2}} = \{remaining\ edges\}$$

We observe that $\chi'(G_i)$=i for i=1,2,...$\frac{n}{2}$

$A\chi'D$ exists for K_n in which the decomposition is minimum.

$AD\chi'(K_n) = \frac{n}{2}$ where n is even

Case ii) When n is odd

First we take $G_1 = \{v_1 v_{n-2}\} \cong K_2$

$$G_2 \cong K_{1,2}$$

$$G_3 \cong C_n$$

$$G_4 \cong K_{1,4}$$

$$\vdots \quad \vdots$$

$$G_{\frac{n+1}{2}} = \{\, remaining\ edges\}$$

We observe that $\chi'(G_i)$=i for i=1,2,... $\frac{n+1}{2}$

$A\chi'D$ exists for K_n in which the decomposition is minimum.

$AD\chi'(K_n) = \frac{n+1}{2}$ where n is odd.

Example 2.19

An $A\chi'D$ $\{G_1, G_2, G_3\}$ of a given graph K_6 is given in Figure 9.

Note that $\chi'(G_i) = i$ for $i = 1,2,3$.

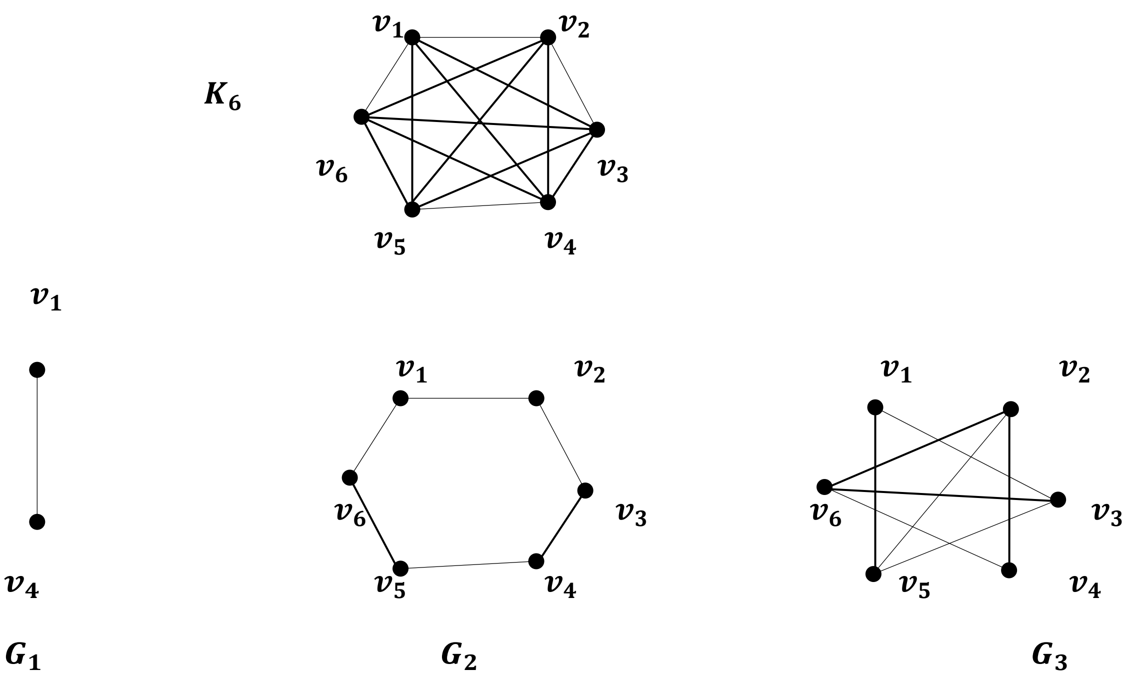

Figure 9: *An* $A\chi'D = \{G_1, G_2, G_3\}$ *of* K_6

Example 2.20

An $A\chi'D$ $\{G_1, G_2, G_3, G_4\}$ of a given graph K_7 is given in Figure 10.

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Note that $\chi'(G_i) = i$ for $i = 1,2,3,4$.

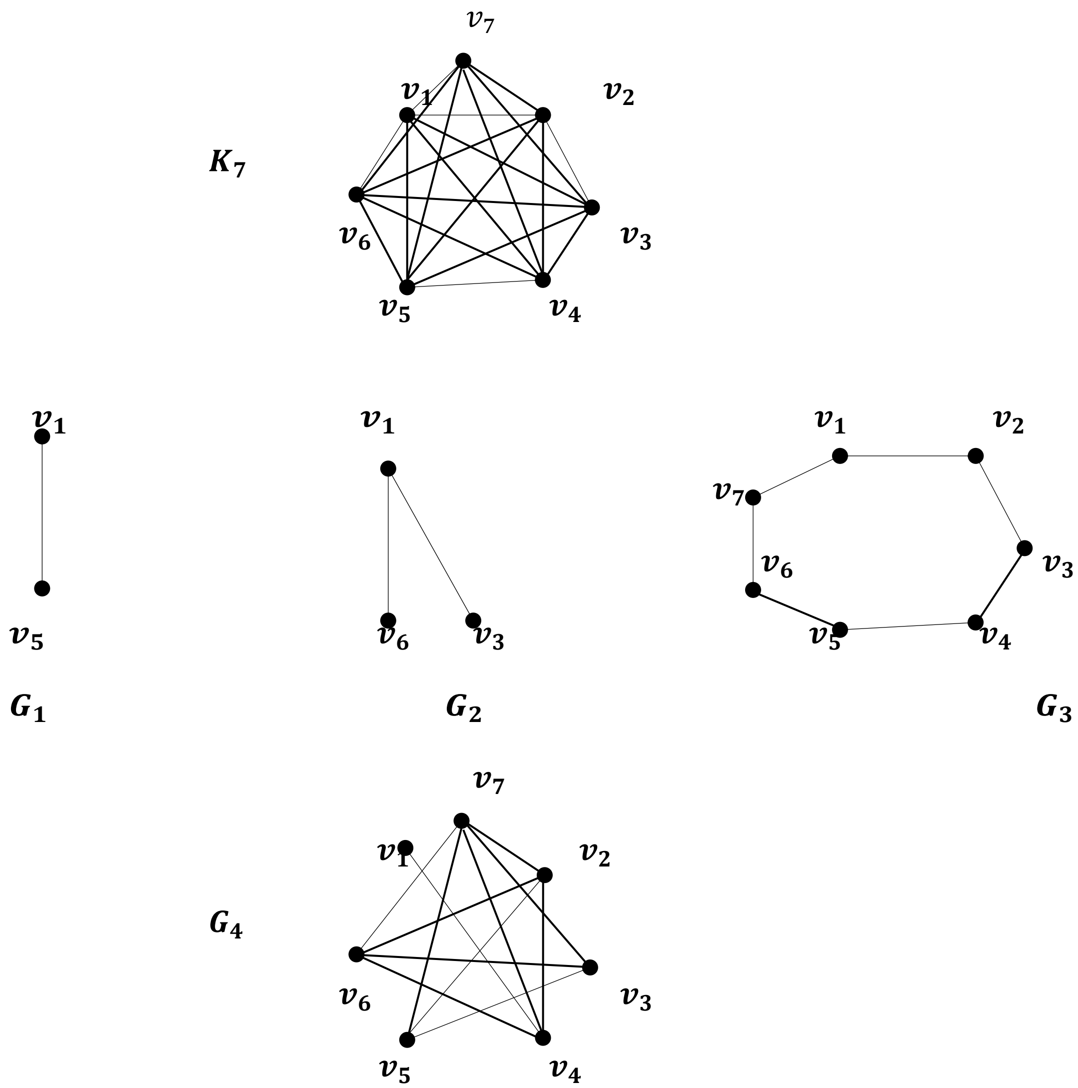

Figure 10: $An\ A\chi'D = \{G_1, G_2, G_3, G_4\}\ of\ K_7$

Theorem 2.21: For any graph G with maximum degree Δ, $AD\chi'$(G) if exists, is at least n, where $n = \frac{-1 \pm \sqrt{1+8\Delta}}{2}$

Proof:

Let v be any vertex in a graph G with maximum degree Δ.

Let $v_1, v_2, \ldots v_\Delta$ be the vertices which are adjacent to v.

Let $G_1 = \{vv_1\}$

$$G_2 = \{vv_2, vv_3\} \cup \textit{the edges which are not adjacent with } v_1, v_2, v_3 \textit{ and of degree} \leq 2$$

$$\vdots \quad \vdots$$

$$G_n = \{vv_l, vv_{l+1}, \ldots, vv_\Delta\}$$
$$\cup \textit{ the edges which are not adjacent with } v, v_l, v_{l+1}, \ldots, v_\Delta \textit{ and of degree} \quad \leq n$$

Clearly, $\chi'(G_i)$=i for i=1,2,… n

We observe that $1+2+\cdots+n=\frac{n(n+1)}{2}=\Delta$ colours.

Therefore, $AD\chi'(\mathrm{G}) \geq \mathrm{n}$ where $\Delta=\frac{n(n+1)}{2}$

$$\Rightarrow 2\Delta= n^2+n$$

$$\Rightarrow n^2+n-2\Delta=0$$

$$\Rightarrow n=\frac{-1\pm\sqrt{1+8\Delta}}{2}$$

Therefore, $AD\chi'(\mathrm{G}) \geq \mathrm{n}$ where $n=\frac{-1\pm\sqrt{1+8\Delta}}{2}$

Corollary 2.22:

For any graph G with maximum degree Δ, $AD\chi'(\mathrm{G})$ exists and is equal to n if $1+8\Delta$ is a odd perfect square.

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